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MAGNETIZED FOKKER-PLANCK EQUATION WITH QUASILINEAR DIFFUSION*

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ABSTRACT

The kinetic equation governing the evolution of a non-axially symmetric velocity distribution function of plasma particles in a uniform magnetic field, including the effects of collisions as well as quasilinear diffusion, is presented and analyzed. It is shown that, if the test particles' orbits are straight lines, the kinetic equation reduces to a standard Fokker-Planck form with friction, collisional diffusion, and quasilinear diffusion coefficients all expressible in terms of scalar potentials. These potentials are simple generalizations of Rosenbluth potentials and can easily be used in numerical solutions of the Fokker-Planck equation.

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I. INTRODUCTION

Current theoretical problems connected with wave heating experiments in both Tokamaks and Tandem Mirrors[1-6] are usually studied on the basis of a kinetic equation including both collisional effects and quasilinear diffusion. Since these experiments are usually carried out in the presence of a confining magnetic field, it is important to take into account the effects of the magnetic field on collisions and quasilinear diffusion. The most notable expressions for the collision integral and quasilinear diffusion term suitable for magnetized plasmas are those derived by Rostoker[7] and Kennel and Engelmann,[8] respectively. In both derivations, however, it was assumed that the distribution functions were azimuthally symmetric about the direction of the magnetic field. As was first pointed out by Haggerty,[9] although this assumption can be justified in several cases of interest, it cannot be imposed when considering problems of plasma transport transverse to the magnetic field. Azimuthal variations of the distribution function give rise to additional terms in the collisional diffusion tensor (see Sec. II below). That these terms importantly contribute to the coefficients of spatial diffusion across the magnetic field was first shown by Rostoker[10] for the case of Maxwellian field particles and later confirmed and generalized in Ref. 11. A more detailed exposition is given in Ref. 12. The first objective of this paper is to examine the expressions given in Refs. 7 and 8 without making the azimuthal symmetry assumption and to compare the results with recent work on magnetized kinetic theory.[13] The more general kinetic equation is presented in Section II. Alternative forms of the collision and quasilinear diffusion operators are discussed in Appendix B.

Magnetized kinetic equations are, in general, extremely complicated and numerical computations based on them are usually very tedious even for the simplest applications.[14,15] The second objective of the paper is to obtain from the general kinetic equation presented in Section II a much simpler magnetized Fokker-Planck equation with quasilinear diffusion which is more suitable for numerical solutions. The analysis is carried out in Section III. The magnetized collision operator derived in this section for non-axially symmetric velocity distribution function provides a simple generalization of the familiar unmagnetized Fokker-Planck equation[16] and can conveniently be used in the recently developed Fokker-Planck codes,[6,17] in which the effects of the external magnetic field on collisions are usually neglected. The conclusions are given in Section IV.

II. THE KINETIC EQUATION

The time evolution of a nonaxially symmetric distribution function, $f_\alpha(\underline{v}_\alpha,t) = f_\alpha(v_{\alpha\perp}, \phi, v_{\alpha\parallel}, t)$, of plasma particles immersed in a uniform magnetic field in the presence of collective collisions and quasilinear diffusion can be described by a kinetic equation of the form

$$\frac{\partial f_\alpha}{\partial t} = (\mathscr{L}_c + \mathscr{L}_{QL})\, f_\alpha \tag{1}$$

in which the collision operator, $\mathscr{L}_c$, and the quasilinear diffusion operator, $\mathscr{L}_{QL}$, are given by

$$\mathscr{L}_C f_\alpha = R_\alpha \frac{\partial}{\partial \underline{v}_\alpha} \cdot \int \frac{\underline{k} d^3k}{k^4} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} ds \frac{\exp\ (iy_\alpha)}{|\varepsilon|^2} \left\{ \Xi(s)\ \mathrm{Re}\ U\underline{k} \cdot \frac{\partial}{\partial \underline{v}_\alpha^0} \right.$$

$$\left. - i\varepsilon_0 m_\alpha k^2 \varepsilon \right\} f_\alpha \qquad (2)$$

$$\mathscr{L}_{QL} f_\alpha = R_\alpha \frac{\partial}{\partial \underline{v}_\alpha} \cdot \int \underline{k}\Psi(\underline{k},t) d^3k \int_0^{\infty} \exp\ (i\bar{y}_\alpha)\ \underline{k} \cdot \frac{\partial f_\alpha}{\partial \underline{v}_\alpha^0}\, ds \qquad (3)$$

where

$$R_\alpha = e_\alpha^2/\varepsilon_\theta^2 m_\alpha^2 (2\pi)^4$$

$$\Xi(s) = 1 + \mathrm{sgn}\ (s)$$

$$U(\underline{k},\omega) = \sum_\beta n_\beta e_\beta^2 \int d^3v_\beta \int_0^{\infty} ds'\ \exp\ (iy_\beta)\ f_\beta(\underline{v}_\beta)$$

$$\varepsilon(\underline{k},\omega) = 1 - \frac{i}{\varepsilon_0 k^2} \sum_\beta \frac{n_\beta e_\beta^2}{m_\beta} \int d^3v_\beta \int_0^{\infty} ds'\ \exp\ (iy_\beta)\ \underline{k} \cdot \frac{\partial f_\beta}{\partial \underline{v}_\beta}$$

$$\Psi(\underline{k},t) = \frac{(2\pi)^2 \exp\ (2\delta t)}{k^4 \left|\frac{\partial \varepsilon(k,\omega)}{\partial \omega}\right|^2_{\omega=p+i\delta}} \sum_{\alpha,\beta} e_\alpha e_\beta n_\alpha n_\beta \int d^3v_\alpha \int d^3v_\beta \int_0^{\infty} ds \int_0^{\infty} ds'$$

$$\exp\ (i\bar{y}_\alpha + i\bar{y}_\beta)\ g_{\alpha\beta}(t = 0)$$

$$y_\alpha = (\omega - k_\parallel v_{\alpha\parallel})s + \frac{k_\perp v_{\alpha\perp}}{\Omega\alpha} \{\sin \nu - \sin (\nu + \Omega_\alpha s)\} \quad , \quad \bar{y}_\alpha = y_\alpha|_{\omega=p+i\delta}$$

$$y_\beta = (\omega - k_\| v_{\beta\|})s' + \frac{k_\perp v_{\beta\perp}}{\Omega_\beta}\{\sin\nu' - \sin(\nu' + \Omega_\beta s')\} \quad , \quad \bar{y}_\beta = y_\beta\big|_{\omega=-p+i\delta} \tag{4}$$

The vector $\underline{k}$, $\underline{v}_\alpha$, $\underline{v}_\alpha^0$, $\underline{v}_\beta$, and v_β^0 are expressed in cylindrical polar coordinates with polar angles χ, ϕ, $\phi + \Omega_\alpha s$, ψ and $\psi + \Omega_\beta s'$ respectively; $\nu = \phi - x$, $\nu' = \psi - x$; $\perp$ and $\|$ denote perpendicular and parallel components with respect to the magnetic field; $\gamma \geq 0$ is the growth rate of the fastest growing mode satisfying the relation $\varepsilon(k, \pm p + i\delta) = 0$; and $g_{\alpha\beta}(t = 0)$ is the initial value of the two particle correlation function.

The expression $\mathscr{L}_c f_\alpha$ was derived from a kinetic theory based on the BBGKY hierarchy of equations[18] and the quasilinear term $\mathscr{L}_{QL} f_\alpha$ has been obtained through a straightforward extension of that theory using the usual quasilinear theory assumptions.[19]

Alternative forms of the operators $\mathscr{L}_c$ and $\mathscr{L}_{QL}$ containing infinite sums of Bessel functions are derived in Appendix A.

III. UNMAGNETIZED TEST PARTICLES

The set of equations (1) through (4) simplify considerably if we neglect the effect of the magnetic field on the test particles and take the limit $\Omega_\alpha \to 0$. In this case $y_\alpha \to (\omega - \underline{k}\cdot\underline{v}_\alpha)s$, $\underline{k} \cdot \partial f_\alpha/\partial \underline{v}_\alpha^0 \to \underline{k} \cdot \partial f_a/\partial \underline{v}_\alpha$ and, after some manipulations, it can easily be shown that the kinetic equation (1) reduces to the standard Fokker-Planck form

$$\frac{\partial f_\alpha}{\partial t} = -\frac{\partial}{\partial \underline{v}_\alpha} \cdot \left(f_\alpha \frac{\partial h_c}{\partial \underline{v}_\alpha}\right) + \frac{1}{2}\frac{\partial}{\partial \underline{v}_\alpha} \cdot \left\{\frac{\partial f_\alpha}{\partial \underline{v}_\alpha} \cdot \frac{\partial^2 (g_c + g_{QL})}{\partial \underline{v}_\alpha \, \partial \underline{v}_\alpha}\right\} \tag{5}$$

in which

$$\frac{\partial h_c}{\partial \underline{v}_\alpha} = \sum_\beta R_{\alpha\beta} \iint \frac{\underline{k} d^3k}{k^4} \int_{-\infty}^{\infty} d\omega \int_0^{\infty} ds' \int_{-\infty}^{\infty} dt$$

$$\exp\,[i(\omega - \underline{k}\cdot\underline{v}_\alpha)t + iy_\beta]\ \underline{k} \cdot \frac{\partial f_\beta}{\partial \underline{v}_\beta^0}\, d^3v_\beta \qquad (6)$$

$$\frac{\partial^2 g_c}{\partial \underline{v}_\alpha\, \partial \underline{v}_\alpha} = 2 \sum_\beta R_{\alpha\beta} \frac{m_\beta}{m_\alpha} \iint \frac{\underline{k}\ \underline{k}}{k^4} d^3k \int_{-\infty}^{\infty} d\omega \int_0^{\infty} ds' \int_{-\infty}^{\infty} dt$$

$$\exp\,[i(\omega - \underline{k} \cdot \underline{v}_\alpha)\ t + iy_\beta]\ f_\beta\ d^3v_\beta \qquad (7)$$

$$\frac{\partial^2 g_{QL}}{\partial \underline{v}_\alpha\, \partial \underline{v}_\alpha} = 2\ R_\alpha \int d^3k\ \underline{k}\ \underline{k}\ \Psi(k,t) \int_0^{\infty} \exp\,[i(p + i\delta - \underline{k}\cdot\underline{v}_\alpha)s]\ ds \qquad (8)$$

where

$$R_{\alpha\beta} = \frac{e_\alpha^2\ e_\beta^2\ n_\beta}{\varepsilon_0^2 (2\pi)^4 m_\alpha m_\beta}$$

The scalar potentials h_c, g_c, and g_{QL} can be derived by integrating (6) through (8) over $\underline{v}_a$ and one readily obtains[20]

$$h_c = +i \sum_\beta R_{\alpha\beta} \iint \frac{d^3k}{k^4} \int_{-\infty}^{\infty} d\omega \int_0^{\infty} ds' \int_{-\infty}^{\infty} \frac{dt}{t}\ \frac{\exp\,\{i(\omega - \underline{k}\cdot\underline{v}_\alpha)t\}}{|\varepsilon|^2}$$

$$\exp\,(iy_\beta)\ \underline{k} \cdot \frac{\partial f_\beta}{\partial \underline{v}_\beta^0}\, d^3v_\beta \qquad (9)$$

$$g_c = 2 \sum_\beta R_{\alpha\beta} \frac{m_\beta}{m_\alpha} \iint \frac{d^3k}{k^4} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} ds' \int_{-\infty}^{\infty} \frac{dt}{t^2} \frac{[1 - \exp\{i(\omega - \underline{k}\cdot\underline{v}_\alpha)t\}]}{|\varepsilon|^2}$$

$$\exp(iy_\beta)\, f_\beta\, d^3v_\beta \tag{10}$$

$$g_{QL} = 4 R_\alpha \int \Psi(k,t)\, d^3k \int_0^\infty \frac{ds}{s^2} [1 - \cos(p - \underline{k}\cdot\underline{v}_\alpha)s] \exp(-\delta s)\, ds$$

$$= 4 R_\alpha \int \Psi(k,t) \left\{ \frac{\delta}{2} \ln\left(\frac{\delta^2}{\delta^2 + (p - \underline{k}\cdot\underline{v}_\alpha)^2}\right)\right.$$

$$\left. + (p - \underline{k}\cdot\underline{v}_\alpha) \tan^{-1}\left(\frac{p - \underline{k}\cdot\underline{v}_\alpha}{\delta}\right)\right\} d^3k \tag{11}$$

Further simplifications of the potentials h_c and g_c can be obtained if we neglect "collective effects" and replace $\varepsilon(k,\omega)$ by its static limit $\varepsilon(k,0) = 1 + k_D^2/k^2$. Thus, upon expressing d^3k in spherical polar coordinates, $d^3k = k^2 \sin\theta\, d\theta\, dk\, d\chi$, one may be able to evaluate the integrals over ω, t, χ, and θ exactly. The algebraic manipulation is carried out in Appendix B and the results are

$$h_c = -8\pi^2 \sum_\beta R_{\alpha\beta} \iint_0^{k_0} \frac{k^2 dk}{(k^2 + k_D^2)} \int_0^\infty \frac{dt}{z^2} (\cos z - \frac{\sin z}{z})\, H_\beta f_\beta d^3v_\beta \tag{12}$$

$$g_c = 16\pi^2 \sum_\beta R_{\alpha\beta} \frac{m_\beta}{m_\alpha} \iint_0^{k_0} \frac{dk}{(k^2 + k_D^2)} \int_0^\infty \frac{dt}{t^2} (1 - \frac{\sin z}{z})\, f_\beta d^3v_\beta \tag{13}$$

where

$$z = tk \left\{ v_{\alpha\perp}^2 + v_{\beta\perp}^2 \frac{\sin^2 (\Omega_\beta t/2)}{(\Omega_\beta t/2)^2} - 2v_{\alpha\perp} v_{\beta\perp} \frac{\sin (\Omega_\beta t/2)}{(\Omega_\beta t/2)} \cos (\mu + \Omega_\beta t/2) + U_\parallel^2 \right\}^{1/2} \tag{14}$$

$$H_\beta = U_\parallel \frac{\partial}{\partial v_{\beta\parallel}} - \left\{ \frac{v_\beta}{\Omega_\beta t} \sin \Omega_\beta t - v_{\alpha\perp} \cos (\mu + \Omega_\beta t) \right\} \frac{\partial}{\partial v_\beta} + \left\{ \frac{\sin^2 (\Omega_\beta t/2)}{\Omega_\beta t/2} - \frac{v_{\alpha\perp}}{v_{\beta\perp}} \sin (\mu + \Omega_\beta t) \right\} \frac{\partial}{\partial \psi} \tag{15}$$

The Fokker-Planck equation (5) with the potentials h_c, g_c, and g_{QL} given by (12), (13), and (11) present the general results of this section. If f_β is independent of ψ, h_c and g_c reduce to the previous results.[21] Furthermore, it is easily established that in the limit of vanishing magnetic field (ie$\Omega_\beta \to 0$), $z \to ktU$, $H_\beta \to -U\nabla' U\cdot\nabla'$ where $\nabla' \equiv \hat{\underline{z}}\, \partial/\partial v_{\beta\parallel} + \hat{\underline{v}}_{\beta\perp} \partial/\partial v_{\beta\perp} + \hat{\underline{\psi}}/v_{\beta\perp}\, \partial/\partial\psi$, and (12) and (13) reproduce the familiar Rosenbluth potentials

$$h_c^0 = 4\pi^3 \sum_\beta R_{\alpha\beta}\, \ell n\, \Lambda \int \frac{f_\beta}{U}\, d^3 v_\beta \tag{16}$$

$$g_c^0 = 4\pi^3 \sum_\beta \frac{m_\beta}{m_\alpha} R_{\alpha\beta}\, \ell n\, \Lambda \int U f_\beta d^3 v_\beta \tag{17}$$

Although the integrals over k and t in (12) and (13) are not amenable to exact analytic treatment, they can be evaluated approximately in closed asymptotic form.[22] The results are

$$h_c = 4\pi^3 \sum_\beta R_{\alpha\beta} \int \left\{ \frac{1}{U} [\ln \Lambda - E_i(\alpha) + E_i(\beta)] + \frac{v_{\alpha\perp}^2}{2q^3} [E_i(\tilde{\alpha}) - E_i(\tilde{\beta})] \right\} f_\beta d^3 v_\beta \tag{18}$$

$$g_c = 4\pi^3 \sum_\beta R_{\alpha\beta} \int \left\{ U[\ln \Lambda - E_i(\alpha) + E_i(\beta)] + q[E_i(\tilde{\alpha}) - E_i(\tilde{\beta})] \right\} f_\beta d^3 v_\beta \tag{19}$$

where E_i is the exponential integral,

$$\alpha = \frac{k_D U}{\Omega_\beta} \ , \quad \beta = \frac{k_0 U}{\Omega_\beta} \ , \quad \tilde{\alpha} = \frac{k_D q}{\Omega_\beta} \ , \quad \tilde{\beta} = \frac{k_0 q}{\Omega_\beta} \ , \quad q = \left(v_{\alpha\perp}^2 + U_\parallel^2 \right)^{1/2}$$

and $\Lambda = k_0/k_D$. It was shown in Ref. 22 that (18) and (19) reduce to the results of Ref. 20 if we take f_β to be Maxwellian and integrate over $d^3 v_\beta$.

IV. CONCLUSIONS

We have presented and discussed in Section II the appropriate magnetized kinetic equation for nonaxially symmetric distribution functions. Both collisional effects and quasilinear diffusion are included. The coefficient of collisional diffusion and quasilinear diffusion possess terms involving the principal value $P(1/\omega - d_\alpha^n)$. These terms disappear only if the distribution functions are azimuthally symmetric about the magnetic field <u>or</u> if the test particles' orbits are straight lines. In the latter case we have shown in Section III that the kinetic equation reduces to a simple magnetized Fokker-Planck equation in which the coefficients of friction, collisional diffusion, and quasilinear diffusion are <u>all</u> expressible in terms of <u>three</u>

scalar potentials. The potentials are simple in form and easily permit numerical computations to be based on them.

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APPENDIX A: ALTERNATIVE FORMS OF THE COLLISION AND QUASILINEAR DIFFUSION OPERATORS

Alternative forms of the operators $\mathscr{L}_c$ and $\mathscr{L}_{QL}$ containing infinite sums of Bessel functions may be derived from Eqs. (2) and (3) upon using the familiar expansions

$$\exp\,(ia \sin\theta) = \sum_{n=-\infty}^{\infty} J_n(a) \exp\,(in\theta)$$

$$\cos\theta \exp\,(ia \sin\theta) = \sum_{n=-\infty}^{\infty} \frac{n}{a} J_n(a) \exp\,(in\theta)$$

$$\sin\theta \exp\,(ia \sin\theta) = -\,i \sum_{n=-\infty}^{\infty} J_n'(a) \exp\,(in\theta) \tag{A1}$$

and the results

$$\underline{k} \cdot \frac{\partial f}{\partial \underline{v}_\alpha^0} = k_\parallel \frac{\partial f_\alpha}{\partial v_{\alpha\parallel}} + k_\perp \cos\,(\nu + \Omega_\alpha s) \frac{\partial f_\alpha}{\partial v_{\alpha\perp}} - \frac{k_\perp}{v_{\alpha\perp}} \sin\,(\nu + \Omega_\alpha s) \frac{\partial f_\alpha}{\partial \phi}$$

$$\int_{-\infty}^{\infty} \exp\,[is(\omega - k_\parallel v_{\alpha\parallel} - n\Omega_\alpha)]\; \Xi(s)\, ds = 4\pi\delta_+\,(\omega - d_\alpha^n) \tag{A2}$$

where

$$\delta_+(x) = \frac{1}{2}\,\delta(x) - \frac{1}{2\pi i x}$$

and

$$d_\alpha^n = k_\parallel v_{\alpha\parallel} + n\Omega_\alpha \quad .$$

Substituting (A1) and (A2) into equations (2) and (3) we find

$$\mathscr{L}_c f_\alpha = 2\pi R_\alpha \sum_{n,m} \int \frac{d^3k}{k^4}\, O_\alpha^m \exp\,[i\nu(m-n)]\; J_m(a_\alpha) J_n(a_\alpha) \int_{-\infty}^{\infty}$$

$$\frac{d\omega}{|\varepsilon|^2}\,\{2R_e U \delta_+(\omega - d_\alpha^n)\; O_\alpha^n f_\alpha - i\varepsilon_0 m_\alpha k^2 \varepsilon \delta(\omega - d_\alpha^n)\} \tag{A3}$$

$$\mathscr{L}_{QL} f_\alpha = iR_\alpha \sum_{n,m} \int d^3k\; \Psi(k,t)\; O_\alpha^m \left\{ \frac{\exp\,[i\bar{\nu}(m-n)]\; J_m(a_\alpha) J_n(a_\alpha) O_\alpha^n f_\alpha}{(i\delta + p - k_\parallel v_{\alpha\parallel} - n\Omega_\alpha)} \right\} \tag{A4}$$

where

$$O_\alpha^n = k_\parallel \frac{\partial}{\partial v_{\alpha\parallel}} + \frac{k_\perp n}{a_\alpha}\, \frac{\partial}{\partial v_{\alpha\perp}} + \frac{ik_\perp}{v_{\alpha\perp}}\, \frac{J_n'(a_\alpha)}{J_n(a_\alpha)}\, \frac{\partial}{\partial\phi} \quad ,$$

$$O_\alpha^m = O_\alpha^n \Big|_{n=m} \text{ and } a_\alpha = \frac{k_\perp v_{\alpha\perp}}{\Omega_\alpha} \quad .$$

If ε, U, and Ψ are azimuthally symmetric about the direction of the magnetic field, the χ integration becomes trivial and (A3) and (A4) reduce to

$$\mathscr{L}_C f_\alpha = 2\pi R_\alpha \sum_n \int \frac{d^3k}{k^4}\, O_\alpha^n J_n^2(a_\alpha) \int_{-\infty}^{\infty} \frac{d\omega}{|\varepsilon|^2} \Big\{ 2R_e U\delta_+(\omega - d_\alpha^n)\, O_\alpha^n f_\alpha$$

$$- i\varepsilon_0 m_\alpha k^2 \varepsilon\delta(\omega - d_\alpha^n)\, f_\alpha \Big\} \qquad \text{(A5)}$$

$$\mathscr{L}_{QL} f_\alpha = iR_\alpha \sum_n \int d^3k\, \Psi(k,t)\, O_\alpha^n \left\{ \frac{J_n^2(a_\alpha)\, O_\alpha^n f_\alpha}{(i\delta + p - k_\parallel v_{\alpha\parallel} - n\Omega_\alpha)} \right\} \qquad \text{(A6)}$$

The result (A5) is in agreement with that derived in Ref. 10 for the case of Maxwellian field particles. Recently, Matsuda[13] derived a similar collision integral using the formalism of Thompson and Hubbard.[23] However, the final result obtained by Matsuda for the case of a quiescent plasma does not include the terms involving P $(1/\omega - d_\alpha^n)$, the principal value part of δ_+. As pointed out in Ref. 10, these terms drop out only if f_α is azimuthally symmetric about the magnetic field. In this case only the real part of δ_+ and the imaginary part of ε contribute to $\mathscr{L}_C f_\alpha$ and (A3) reduces to the famous magnetized collision integral.[7] Furthermore, if f_α is independent of ϕ, Eq. (A4) reduces to the familiar quasilinear diffusion term.[8]

APPENDIX B: EVALUATION OF INTEGRALS

We carry out here the integrals over ω, t, χ, and θ in equations (9) and (10).

If we approximate $\varepsilon(k,\omega)$ by its static value $\varepsilon(k,0) = 1 + k_D^2/k^2$, we can trivially perform the integrals over ω and t to get

$$h_c = 2\pi i \sum_\beta R_{\alpha\beta} \iint \frac{d^3k}{(k^2 + k_D^2)^2} \int_0^\infty \frac{ds'}{s'} \underline{k} \cdot \frac{\partial f_\beta}{\partial v_\beta^0} \exp i(y_\beta - \underline{k}\cdot\underline{v}_\alpha s') \tag{B1}$$

$$g_c = 4\pi \sum_\beta \frac{m_\beta}{m_\alpha} R_{\alpha\beta} \iint \frac{d^3k}{(k^2 + k_D^2)^2} \int_0^\infty \frac{ds'}{s'^2} \{1 - \exp i(y_\beta - \underline{k}\cdot\underline{v}_\alpha s')\} \tag{B2}$$

The integral over χ is next to be evaluated. The exponent in (B1) and (B2) can be expressed in the form

$$y_\beta - \underline{k}\cdot\underline{v}_\alpha s' = s' k_\parallel U_\parallel + (A \cos\nu + B \sin\nu) \tag{B3}$$

where

$$U_\parallel = v_{\beta\parallel} - v_{\alpha\parallel} \ ,$$

$$A = s' k_\perp \left\{\frac{v_{\beta\perp}}{\Omega_\beta t/2} \sin\frac{\Omega_\beta s'}{2} \cos\left(\mu + \frac{\Omega_\beta s'}{2}\right) - v_{\alpha\perp}\right\} \ ,$$

$$B = -\frac{2k_\perp v_{\beta\perp}}{\Omega_\beta} \sin\frac{\Omega_\beta s'}{2} \sin\left(\mu + \frac{\Omega_\beta s'}{2}\right) \ ,$$

and

$$\mu = \psi - \phi \quad .$$

Thus, writing $d^3k = k^2 \sin\theta d\theta\, dk\, d\chi$, changing the variable of integration from χ to $\nu = \phi - \chi$ and making use of the integral

$$\int_0^{2\pi} \exp\,(iA\cos\nu + iB\sin\nu)\,d\nu = 2\pi\, J_0[(A^2 + B^2)^{1/2}] \quad ,$$

equations (B1) and (B2) reduce to, after performing the ν integration,

$$h_c = 4\pi^2 i \sum_\beta R_{\alpha\beta} \int_0^{k_0} \frac{k^2 dk}{(k^2 + k_D^2)^2} \int_0^\pi \sin\theta\, d\theta \int_0^\infty \frac{ds'}{s'} \exp\,(is'k_\parallel U_\parallel)$$

$$\Bigg[J_0\,[(A^2 + B^2)^{1/2}]\, k_\parallel \frac{\partial}{\partial v_{\beta\parallel}} + \frac{ik_\perp J_1[(A^2 + B^2)^{1/2}]}{(A^2 + B^2)^{1/2}} \{A\cos\,(\mu + \Omega_\beta s')$$

$$- B\sin\,(\mu + \Omega_\beta s')\} \frac{\partial}{\partial v_{\beta\perp}} - \frac{ik_\perp J_1[(A^2 + B^2)^{1/2}]}{v_{\beta\perp}(A^2 + B^2)^{1/2}} \{B\cos\,(\mu + \Omega_\beta s')$$

$$+ A\sin\,(\mu + \Omega_\beta s')\} \frac{\partial}{\partial\psi} \Bigg] f_\beta\, d^3v_\beta \tag{B4}$$

$$g_c = 8\pi^2 \sum_\beta \frac{m_\beta}{m_\alpha} R_{\alpha\beta} \int_0^{k_0} \frac{k^2 dk}{(k^2 + k_D^2)^2} \int_0^\pi \sin\theta\, d\theta \int_0^\infty \frac{ds'}{s'^2} \exp\,(is'k_\parallel U_\parallel)$$

$$\Big[1 - J_0\,[(A^2 + B^2)^{1/2}] \Big] f_\beta d^3v_\beta \tag{B5}$$

where k_0 is the usual cutoff at the inverse of the distance of closest approach.

The θ integrals in (B4) and (B5) may easily be reduced to the evaluation of the integral

$$\int_0^{\pi} \exp\,(iq\cos\theta)\, J_0\,(p\sin\theta)\,\sin\theta\, d\theta = 2\,\frac{\sin\,(p^2+q^2)^{1/2}}{(p^2+q^2)^{1/2}} \quad . \qquad \text{(B6)}$$

Thus using (B6) to carry out the integration over θ we finally arrive at the results (12) and (13).

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